Relativity in Global Satellite Navigation Systems

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1. Navigation—why you need a clock
2. Brief history of relativity in the GPS
3. What the GPS is
4. Relativistic effects:
   Relativity of synchronization;
   Time dilation;
   Gravitational frequency shifts;
   Sagnac effect;
5. Observations: testing relativity
   TOPEX;
   Frequency jumps;
   Unmodeled effects;
6. Applications
Latitude
Moon, Jupiter & Satellites
GPS RELATIVITY MEETINGS

• 1979  SAMSO Relativity Seminar (Boulder)
• 1985  JASON Study
• 1986  Air Force Studies Board
• 1988-98 Various Working Group Meetings
• 1995  ARL-Chapel Hill
• 1997  ICD-200 Relativity Review (Boulder)
GPS RELATIVITY MEETINGS

- 1979: SAMSO Relativity Seminar
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Erroneous Reports

- 1977-83
- 1992
- 1996
- 2000-2006
- 2008
GPS Constellation

- 24 Satellites (now > 30)
- 6 orbital planes, 55° inclination
- Period: half a sidereal day
- Several atomic clocks/satellite
- Several spare satellites
Control Segment

GPS Control Segment

- Master Control Station
- Alternate Master Control Station
- Ground Antenna
- AFSCN Remote Tracking Station
- Air Force Monitor Station
- NGA Monitor Station

Updated April 2014
GPS IIR Satellite
Block III satellite
Block III GPS satellite
Other GNSS Satellites

Beidou

GALILEO
## Constellation Status-GPS

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<th>Slot</th>
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### GPS Constellation Status

SUBJ: GPS STATUS  11 JUN 2017

1. SATELLITES, PLANES, AND CLOCKS (CS=CESIUM RB=RUBIDIUム):

   A. BLOCK I: NONE

   B. BLOCK II: PRNS  1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15

   PLANE : SLOT D2, D1, E1, E3, D4, A4, C3, F3, E2, D5, B4, F6, F1, F2

   CLOCK : RB, RB, RB, RB, RB, CS, RB, RB, RB, RB, RB, RB, RB

   BLOCK II: PRNS 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29

   PLANE : SLOT B1, C4, E4, C5, B6, D3, E6, F4, A1, B2, B5, C2, B3, C1

   CLOCK : RB, RB, RB, RB, RB, RB, CS, RB, RB, RB, RB, RB, RB

   BLOCK II: PRNS 30, 31, 32

   PLANE : SLOT A3, A2, F5

   CLOCK : RB, RB, RB
GALILEO Constellation Status 11/21/2016

**Reference Constellation Orbital and Technical Parameters**

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<th>Excentricity</th>
<th>Inclination (deg)</th>
<th>RAAN (deg)</th>
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**Extended Slots**

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<th>Semi-Major Axis (km)</th>
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Planned Beidou Constellation

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*Final BeiDou Constellation*
Beidou Constellation Status

As of November 2016: 20 operational satellites of 35 planned

- 6 satellites in geostationary orbits;
- 8 in 55-degree inclined geosynchronous orbits;
- 6 in medium earth orbits at altitude 21,500 km
Fundamental Principles

• **Constancy of the speed of light**
  – The speed of light, $c$, is a constant independent of the motion of the source (or of the observer);

• **Principle of Equivalence (“weak form”)**
  – Over a small region of space and time, the fictitious gravitational field induced by acceleration cannot be distinguished from a real gravitational field due to a mass.
Constancy of $c$

$$| \mathbf{r} - \mathbf{r}_j | = c(t - t_j) ,$$

$j = 1, 2, 3, 4$

Synchronization Is the key!
Reciprocity

\[ |\mathbf{r}_j - \mathbf{r}| = c(t_j - t) \]

Detector j; \ j=1,2,3,4
Exploded Block IIR Satellite View
Clock Improvement Since 1000 A.D.

1 ns/day = $10^{-14}$

Timekeeping Performance

Year - A.D.

Time Stability in Seconds per Day

0.1 ps
1 ps
10 ps
100 ps
1 ns
10 ns
100 ns
1 µs
10 µs
100 µs
1 ms
10 ms
100 ms
1 s
10 s
100 s
1 ks
10 ks
Why are atomic clocks needed?

To reduce the effect of clock error to < 2 meters,

the clock error must be less than \( \frac{2}{c} = 6.7 \times 10^{-9} \) sec.

Half a day = 43200 seconds, so the fractional clock error must be less than:

\[
\frac{2 \text{ m}}{43200 \text{ s} \times c} = 1.5 \times 10^{-13}.
\]

Only atomic clocks can achieve such stability.
Frequency Stability of Navstar Block IIR/IIRM Rubidium Timing Signal Offset from Washington DC Time Reference 1-JAN-08 to 1-JUL-08

Figure 2-7. Frequency Stability Profiles (Hadamard) of Block IIR/IIR-M Rubidium Clocks
Constancy of the speed of light implies time dilation

Thought experiment viewed in “moving” frame.

(These clocks are at rest in the moving frame.)

Thought experiment viewed in “rest” frame.

(This clock at rest in “rest” frame, coincides with upper clock in moving frame.)
Einstein’s Light Clock

Time = \frac{\text{Distance}}{c}

\sqrt{c^2 - v^2}

\begin{align*}
t &= \frac{L}{\sqrt{c^2 - v^2}} = \frac{L}{c} \frac{1 - v^2}{c^2} \\
t' &= \frac{L}{c} = \sqrt{1 - \frac{v^2}{c^2}} t
\end{align*}
How Big is Time Dilation?

\[ \sqrt{1 - \frac{v^2}{c^2}} \approx 1 - \frac{1}{2} \frac{v^2}{c^2}; \]

\( v = 4000 \text{ m/s}; \)

\[ -\frac{1}{2} \frac{v^2}{c^2} = -8.9 \times 10^{-11} \]

(about 8 microseconds per day)
Accounting For Relativistic Effects

Example: Time Dilation: 

\[ d\tau = \sqrt{1 - \frac{v^2}{c^2}} \, dt; \]

\[ dt = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \, d\tau \]

\[ \equiv \left(1 + \frac{1}{2} \frac{v^2}{c^2}\right) d\tau. \]

Elapsed Coordinate time: 

\[ \Delta t = \int_{\text{path}} d\tau \left(1 + \frac{1}{2} \frac{v^2}{c^2}\right) \]

Observed Proper Time

Note: 

\[ (cd\tau)^2 = (cdt)^2 (1 - \frac{v^2}{c^2}) = (cdt)^2 - dx^2 - dy^2 - dz^2 \]
For light: \[ 0 = ds^2 = (cdt)^2 - dx^2 - dy^2 - dz^2 \]

Time dilation: \[ ds^2 = (cd\tau)^2 = (cdt)^2 \left( 1 - \frac{1}{c^2} \frac{dx^2 + dy^2 + dz^2}{dt^2} \right) \]

\[ = (cdt)^2 - dx^2 - dy^2 - dz^2 \]

With gravity: \[ ds^2 = \left( 1 + \frac{2\Phi}{c^2} \right)(cdt)^2 - \left( 1 - \frac{2\Phi}{c^2} \right)(dx^2 + dy^2 + dz^2) \]

Motion of Planets: \[ \delta \int_{path} ds = 0. \]
Coordinate Time

• In special relativity:
  To each real clock, corrections are applied such that at each instant, the clock would read the same as a hypothetical clock at rest at the same point in the underlying inertial frame.

• When gravitational fields are present:
  Additional corrections compensate for gravitational frequency shifts relative to a reference on earth’s geoid.

• GPS time is an example of coordinate time, in which the reference is on the earth’s rotating geoid.
Sagnac Effect

\[(c d \tau)^2 = ds^2 = (c dt)^2 - dr^2 - r^2 (d \phi_{ECI})^2 - d\tilde{z}^2\]

In a rotating coordinate system such as one fixed to the earth, let the axis representing the zero for the angle \( \phi \) rotate with constant angular speed:

\[ds^2 = (c dt)^2 - dr^2 - r^2 (d(\phi - \omega t))^2 - d\tilde{z}^2\]

\[= \left(1 - \frac{\omega^2 r^2}{c^2}\right)(c dt)^2 + 2\omega r^2 d\phi dt - dr^2 - r^2 d\phi^2 - d\tilde{z}^2\]

This is the Langevin metric.

For light: solving for dt to first order in \( \omega \), the \( d\phi dt \) term gives rise to the Sagnac effect.

\[dt = \frac{d\sigma}{c} + \frac{\omega r^2 d\phi}{c^2}\]
Sagnac Correction = \( \frac{2\omega}{c^2} A_z \)
Over a small region of space and time, a fictitious gravity field induced by acceleration cannot be distinguished from a gravity field produced by mass.

Equivalence principle and gravitational frequency shifts
Gravitational Frequency Shift
Gravitational Frequency Shift

\[ t = \frac{L}{c}; \]

\[ v = gt = \frac{gL}{c}; \]

\[ \frac{\Delta f}{f} = -\frac{v}{c} = -\frac{gL}{c^2} = -\frac{\Delta \Phi}{c^2}. \]
Relativity of Simultaneity

To an observer on the ground, let two lightning strokes at the front and back of the train be simultaneous.

The “moving” observer at the train’s midpoint finds the event at front occurs first.

\[ t' \approx t - \frac{vx}{c^2} \Rightarrow t - \frac{V \cdot r}{c^2} \]
Induced potential difference/c² = \frac{A \cdot r}{c²} = \frac{\Delta V \cdot r}{\Delta Tc^2};

Gravitational potential difference/c² = \frac{\nabla\Phi \cdot r}{c²};

Net potential difference/c² = \frac{A \cdot r + \nabla\Phi \cdot r}{c²} = 0;
Fundamental Scalar Invariant

\[
(c \, d\tau)^2 = \left(1 + \frac{2\Phi}{c^2}\right)(c \, dt)^2 - \left(1 - \frac{2\Phi}{c^2}\right)(dx^2 + dy^2 + dz^2)
\]

\[
\Phi = -\frac{GM}{r} \left(1 - \frac{J_2 a_1^2}{r^2} \left[\frac{3z^2}{r^2} - \frac{1}{2}\right]\right)
\]

For a clock near earth,

\[
\Delta t = \int_{\text{path}} d\tau \left[1 + \frac{1}{2} \frac{v^2}{c^2} - \frac{\Phi}{c^2}\right]
\]
Earth-fixed Clock

\[
\Delta t = \int \limits_{\text{path}} d\tau \left[ 1 - \frac{\Phi_0}{c^2} \right];
\]

\[
\frac{\Phi_0}{c^2} = -\frac{GM}{c^2 a_1} - \frac{GMJ_2}{2c^2 a_1} - \frac{\omega^2 a_1^2}{2c^2}
\]

\[
= (-6.95348 - .00376 - .01203) \times 10^{-10}
\]

\[
= -6.96927 \times 10^{-10}
\]

This is the fractional frequency shift of an atomic clock fixed on earth, relative to an atomic clock at infinity.

Note about centripetal term
Clocks on earth’s geoid beat at equal rates.

Clocks at rest on geoid beat at equal rates, defining International Atomic Time. They are synchronized in the underlying inertial frame.

Centripetal potential, monopole potential, quadrupole, and higher potential terms conspire to give an equipotential in the rotating frame.
Earth-based Time Scale

\[ L_G \equiv \frac{-\Phi_0}{c^2} \equiv 6.969290134 \times 10^{-10}. \] (This number is now a defined quantity.)

**SI Second:**

\[ t \Rightarrow \left( 1 - \frac{\Phi_0}{c^2} \right) t \]

\[ (c \, d\tau)^2 = \left( 1 + \frac{2(\Phi - \Phi_0)}{c^2} \right) (c \, dt)^2 - \left( 1 - \frac{2\Phi}{c^2} \right) (dx^2 + dy^2 + dz^2) \]

(Basis for International Atomic Time, Universal Coordinated Time.)
Atomic Clock in a Satellite

\[ \frac{1}{2} v^2 + \Phi = -\frac{GM}{2a}; \quad a = \text{semimajor axis} \]

\[ d\tau = \left(1 + \frac{2(\Phi - \Phi_0)}{c^2} - \frac{v^2}{c^2}\right)^{1/2} \, dt \quad \text{or} \quad \frac{dt}{d\tau} = 1 - \frac{(\Phi - \Phi_0)}{c^2} + \frac{v^2}{2c^2} \]

\[ \Delta t_{SV} = \int_{\text{path}} d\tau \left[ 1 + \frac{3GM}{2ac^2} + \frac{\Phi_0}{c^2} - \frac{2GM}{c^2} \left( \frac{1}{a} - \frac{1}{r} \right) \right] \]

\[ = \int_{\text{path}} d\tau [1 - 4.4647 \times 10^{-10}] \quad -38.6 \mu s/\text{day} \]

\[ + \frac{2\sqrt{GM}}{c^2} e^{\sqrt{\frac{a}{\text{meter}}} \sin E} \, (\text{sec}) \]
Factory Frequency Offsets

<table>
<thead>
<tr>
<th>GNSS System</th>
<th>a (km)</th>
<th>$10^{12} \times \Delta f/f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>GLONASS</td>
<td>25509.64</td>
<td>-436.144</td>
</tr>
<tr>
<td>GPS</td>
<td>26562.76</td>
<td>-446.473</td>
</tr>
<tr>
<td>BEIDOU (MEO)</td>
<td>27910.20</td>
<td>-458.538</td>
</tr>
<tr>
<td>GALILEO</td>
<td>29601.31</td>
<td>-472.191</td>
</tr>
<tr>
<td>Geosynchronous</td>
<td>42164.17</td>
<td>-539.151</td>
</tr>
</tbody>
</table>
Net fractional frequency shift of a clock
in a circular orbit
relative to a reference on the rotating geoid

Frequency shifts cancel at this radius
Three Important Effects (GPS)

#1: Scale correction to satellite clock:

\[
10.23000000000 \text{ MHz} \rightarrow 10.22999999543 \text{ MHz}
\]

#2: Receiver must implement the eccentricity correction:

\[
+ 4.4428 \times 10^{-10} e \sqrt{\frac{a}{\text{meter}}} \sin E \text{ (sec)}
\]

#3: User must account for time required for signal propagation (Sagnac effect) if relevant.
SV#13 eccentricity effect $e = 0.013$

(TOPEX receiver)

Time from beginning of day Oct 22 1995 (s)
GALILEO Satellites in unintended orbits

<table>
<thead>
<tr>
<th>Extended Slots</th>
<th>GSAT0201</th>
<th>18</th>
<th>Ext01</th>
<th>2014-08-22</th>
<th>27977.6</th>
<th>0.162</th>
<th>49.850</th>
<th>52.521</th>
<th>56.198</th>
<th>316.069</th>
</tr>
</thead>
<tbody>
<tr>
<td>GSAT0202</td>
<td>14</td>
<td>Ext02</td>
<td>2014-08-22</td>
<td>27977.6</td>
<td>0.162</td>
<td>49.850</td>
<td>52.521</td>
<td>56.198</td>
<td>136.069</td>
<td></td>
</tr>
</tbody>
</table>

Normal radius of a GALILEO satellite: 29599.8 km
Eccentricity: 0
Quasi-Zenith Satellite System (Japan)

Relativistic effect on satellite clock time
QZSS geosynchronous orbit, $e = 0.075$
Frequency “Breaks”
Due to Orbit Adjustments

\[
\frac{\Delta f}{f} = -\frac{3GM}{2c^2a} - \frac{\Phi_0}{c^2} + \frac{2GM}{c^2a} \left[ \frac{1}{r} - \frac{1}{a} \right].
\]

If eccentricity is small,

\[
\delta \left( \frac{\Delta f}{f} \right) = + \frac{3GM \delta a}{2c^2a^2};
\]

FOR SV#43:

Measured: \(-1.85 \times 10^{-13}\)

Predicted: \(-1.77 \times 10^{-13}\)

Now implemented.
Unmodeled Relativistic Effect: Oblateness

Effect of Earth’s oblateness on satellite orbit:

\[ \delta \Phi = \Phi_{J_2} = \frac{GMJ_2}{r^3} \left( \frac{3z^2 - r^2}{2r^2} \right); \]

Change in monopole potential:

\[ \delta \left( -\frac{GM}{r} \right) = \frac{GMJ_2 a_1^2 \sin^2 I}{4a^3} \cos 2(\omega + f) + ... \]

Change in kinetic energy:

\[ \delta \left( \frac{v^2}{2} \right) = \frac{GMJ_2 a_1^2 \sin^2 I}{2a^3} \cos 2(\omega + f) + ... \]

Change in frequency:

\[ \delta \left( \frac{\Delta f}{f} \right) = -\frac{GMJ_2 a_1^2 \sin^2 I}{a^3 c^2} \cos 2(\omega + f) + ... \]
A Coincidence?

There are many terms in the perturbations arising from Earth’s oblateness with coefficient

\[
\left( 1 - \frac{3}{2} (\sin I)^2 \right)
\]

For GPS, this is nearly zero. \((I = 55^\circ)\)
Shapiro delay SV to earth surface

\[ \Delta t_{\text{gravity}} = -\frac{|r_2 - r_1| L_G}{c} + \frac{G M_e}{c^3} \log \left( \frac{r_1 + r_2 + |r_2 - r_1|}{r_1 + r_2 - |r_2 - r_1|} \right) \]

Fig. 31 Time delay vs elevation angle \( E \) for a GPS satellite-to-user link, including the time scale change for reference clocks on Earth’s surface.
Spectrum of lunar tidal potential

Detailed calculation of the lunar tidal potential gives perturbations in terms of

$$\sum_i A_i \cos(n_i \omega_{sat} t + m_i \omega_{moon} t + \varphi_i)$$

$$n_i = -6, \ldots + 8; \quad m_i = -7, \ldots + 8$$

The coefficients are functions of the eccentricities and inclinations of the SV and the moon with respect to the equator. The phases are functions of the altitudes of perigee and the angles of the lines of nodes.

There are significant contributions from many frequencies in the neighborhood of 6 hours. (These correspond to $n_i = 2$.)

The short-period terms are sufficiently close together that they can beat against each other, reinforcing and cancelling. They can combine and have amplitudes that are estimated to be greater than about $2 \times 10^{-15}$. 
Lunar and solar tidal perturbations are estimated to affect the fractional frequency shifts of GPS SV clocks in a predictable way by about

$$3.7 \times 10^{-15}$$

The principal periods with which this occurs are near 6 hours but there are many nearly equal frequencies.
Surface Plate Velocities
Control of Monster Machines

"When the company put CAES on its bulldozers, we had increases of 30% in productivity. It's not that the operators are moving any more dirt...they're getting it to the right place the first time."

— Joe Long, Black Thunder, Wyoming

Caterpillar's Computer Aided Earthmoving System (CAES) brings a new level of productivity and efficiency to open-cast mining at Black Thunder in Wyoming – North America's largest surface coal mine.

An in-cab display gives operators easy-to-understand line diagrams of where to cut and fill. The system uses on-board computers, software, data radios and Continuously-Operational Surveying System (COPS), readers to continually monitor work and update the plan.

With CAES, the effort of reading maps or looking for grade stakes is virtually eliminated. And, because it gives instantaneous, accurate feedback, operators can dig quickly, accurately and confidently.
Autonomous Operation
Precision Agriculture
Surveying

Finding boundary markers lost for a century.
Animal Tracking

A fully instrumented DGPS sheep
GNSS-other satellite navigation systems

GLONASS-Russia
GALILEO-ESA
BEIDOU--China
IRNSS--INDIA
QZSS--JAPAN

AUGMENTATION SYSTEMS:
WAAS
EGNOS

All use the same fundamental relativity concepts.

The GALILEO specs state “all relativity corrections are the responsibility of the user.”
Clock Coefficient $a_0$ of GPS Satellite clocks, 1992-2014

$$10^{12} \times \frac{\Delta f}{f}$$
References:


“100 Years of Relativity,” World Scientific,
A. Ashtekar, ed., (2005), Chapter 10

“Handbook of Spacetime,” Springer,
Ashtekar and Petkov, eds, (2014), Chapter 24

“General Relativity: The Most Beautiful of Theories,”

END
GPS DEVELOPMENT KIT
CONSENSUS:

1. To meet its design goals, GPS requires the consideration of relativistic effects. These effects are best described by the theory of General Relativity (which includes special relativity), and are fully deterministic. The theory is supported by the small amount of experimental data at an adequate level. Therefore the effects can and must be accounted for to the necessary levels of accuracy. The process of synchronization of clocks for GPS is best understood from a local, geocentric, nonrotating freely falling frame of reference.

2. The presentation by the staff of General Dynamics Electronics Division, as the contracting support organization, indicates that relativistic corrections are currently being implemented in GPS in a manner such that all significant corrections are included.

3. Appendix VI of Document CP-CS 304, Part I Code Ident 12436, 9 January 1978 entitled "Computer Program Development Specifications for the GPS Master Control Station Ephemeris Computer Programs" has significant errors in it and should be disregarded.

4. The relativistic correction due to the Sun (General Dynamics Electronics Division view graph 792X-039) showing an effect "of order 40 cm" is incorrectly calculated; however, this correction is not currently used and the effect is expected to be much smaller.

RECOMMENDATIONS:

1. General Dynamics Electronics Division should update their official documentation to correct errors such as those identified above.

2. The existing documentation is not suitable for communicating to the scientific and engineering community the nature of the relativistic corrections and the procedures by which they are implemented. More appropriate documentation should be prepared.

3. The designers of GPS user equipment in particular should have available adequate and proper documentation so that the appropriate relativistic corrections can be included.